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Stress and Strain: $T^{\mu\nu}$ of Higher Spin Gauge Fields

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Abstract: We present some results concerning local currents, particularly the stress tensors $T^{\mu\nu}$, of free higher (>1) spin gauge fields. While the $T^{\mu\nu}$ are known to be gauge variant, we can express them, at the cost of manifest Lorentz invariance, solely in terms of (spatially nonlocal) gauge-invariant field components, where the "scalar" and "spin" aspects of the systems can be clearly separated. Using the fundamental commutators of these transverse-traceless variables we verify the Poincaré algebra among its generators, constructed from the T^0_μ and their moments. The relevance to the interaction difficulties of higher spin systems is mentioned.

1 Introduction

Gauge fields with spin >1 enjoy a (deservedly) bad reputation, at least when they are not parts of some infinite tower, possibly in nonflat backgrounds. Their best-known problems lie in the difficulty of consistent interaction with gravity, and (with the sole, but significant, exception of self-interaction spin 2, *aka* general relativity) with themselves. Here we give a brief and preliminary discussion of work [1] of a related but rather different aspect of gauge field problems, namely the unavoidable gauge variance of their stress-tensors $T^{\mu\nu} = T^{\nu\mu}$; non-symmetric tensors are uninteresting because they do not define rotation generators. What has been known for some time [2] is that those of their spatially integrated moments corresponding to the Poincaré generators $(P_\mu, J^{\alpha\beta})$ are nevertheless both gauge invariant and obey the Poincaré algebra ensuring the invariance of the underlying theory. More precisely, the result of [2] is that all the gauge dependence of the $T^{0\mu}$ is concentrated in identically conserved superpotential $S^{0\mu} \equiv \partial^2_{\alpha\beta}\Delta^{0\alpha\mu\beta}$, where the Δ have the algebraic symmetries of the Riemann tensor. This means that $S^{00} \equiv \partial^2_{ij}\,\Delta^{0i0j}$ can contribute neither to the energy $\int d^3x\, T^0_0$, nor to the boosts $K^i = \int d^3x\; x^i T^{00}$. Likewise, the effect of $S^{0i} \equiv \partial^2_{k\ell}\Delta^{0ki\ell} + \partial^2_{k0}\,\Delta^{0ki0}$ vanishes both in the momentum $\int d^3x T^0_i$ and in the rotation generators $\int d^3x(x^iT^{0j} - x^jT^{0i})$. [That the first part of S^{0i} does not contribute is obvious; the second doesn't due to the (ik) symmetry of Δ^{0ki0}.] So all free fields in flat space are indeed safe (we will also discuss deSitter backgrounds in [1]). In fact one could finish the discussion right here by noting that as long as a current is

conserved on shell in a field theory, and plays no dynamical role as a "source" then all that need be demanded of it is that its spatial integrals produce gauge invariant generators of the corresponding transformation (here Poincaré rotations). But then free fields are always well-behaved and dull. It is in their role and effects when interaction is introduced that the local currents must come under scrutiny; at the very least they must consistently couple to gravity, so their $T^{\mu\nu}$ do count!

That $T^{\mu\nu}$ is gauge variant for spin >1 (we stick to massless, gauge, fields throughout) is obvious: The form of $T^{\mu\nu}$, in terms of potentials, $\phi_{\alpha\beta\ldots}$ is of course $T \sim a\,\partial\phi\,\partial\phi + b\,\phi\partial^2\phi$; the b term can generally be exchanged for a superpotential. [Correspondingly $T \sim \psi\partial\psi$ for fermions, which face the same problems for $s > \frac{1}{2}$, except that there is one less derivative. We leave this parallel story to the reader.] But for $s > 1$ at least two derivatives are required to define local gauge invariants, namely (linearized) "curvatures". [The Maxwell tensor is safe because the curl of A_μ is already invariant, and $A\partial^2 A$ terms needn't appear.] From this point of view, it seems miraculous that the Poincaré generators are invariant; our approach should also dispel this paradox.

2 Vector Currents

It is perhaps instructive to note first that loss of gauge invariance in currents already occurs at spin 1, where the current is the vector j^μ associated with invariance under internal rotations of a multiplet of vector gauge fields, rather than with the space-time invariances of $T^{\mu\nu}$. The simplest example is a doublet, the complex field C_μ obeying Maxwell's equations. The associated (neutral) current is $j^\mu = (iG^{\mu\nu *}C_\nu + c.c.)$, unique up to superpotentials $\Delta j^\mu \equiv \partial_\nu \Sigma^{\mu\nu}$, $\Sigma^{\mu\nu} = -\Sigma^{\nu\mu}$. The dependence on the potential C_ν, and not only on the field strength $G_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu$, is the source of gauge variance here, and cannot be "improved" away. The total charge $Q = \int d^3x\, j^0$ is (of course) conserved and gauge invariant (on shell) under the local gauge transformation: since $j^0 \sim G^{0i*}C_i$, then $\delta j^0 \to \partial_i(G^{0i*}\,\Lambda)$ under $\delta C_i = \delta_i\,\Lambda$, owing to the Gauss constraint $\partial_i\, G^{0i} = 0$. The "gauge-invariant" form of j^0 is best exhibited in radiation gauge, where A_i, like G^{0i}, is also transverse: $j_0 \sim G^{*T}_{0i}\, A^T_i$, or equivalently $j_0 \sim G^T_{0i}\, A^T_i + \partial_i(G^T_{0i}\, A^L)$. The analogy to stress tensors can be taken one more step. Just as spin 2 can be deformed to GR when coupled to its stress tensor, so does the coupling of complex field above to j^μ deform to become Yang–Mills when a third, neutral, A_μ field is introduced to provide the $j^\mu(C)A_\mu$ interaction [3] and complete the triplet, (C_μ, A_μ).

3 Stress Tensors

Our aim is to exploit the fact that renouncing manifest Lorentz invariance allows us to restate $T^{\mu\nu}$ in "manifestly" gauge-invariant form – making gauge choices to be sure.

The existence of gauge invariant representations of $T^{\mu\nu}$ can be understood as follows. For concreteness, we use D=4, where all $s > 0$ gauge fields have just two independent, helicity $\pm s$, modes, obeying the wave equation. [These theories are formulated in terms of totally symmetric tensors $\varphi_{\mu_1\ldots\mu_s}$ (subject to double tracelessness for $s > 3$).] They correspond (say in symmetric tensor representation) to (spatially nonlocal) transverse-traceless spatial component $\phi^{TT}_{ij\ldots}$. The other field components are constraint variables, Lagrange multipliers or pure gauges. Hence the

original, gauge invariant, action[1] will reduce, upon enforcing the constraints, *but in any gauge,* to the simple "oscillator" form

$$I = \int d^4x \left[\sum_1^2 p_A \dot{q}_A - H(p,q) \right] , \quad H = \frac{1}{2} \left\{ p^2 + (\nabla q)^2 \right\} . \tag{3.1}$$

The two conjugate pairs (p_A, q_A) denote the appropriate spatial TT variables, with implicit summation over indices. While this representation of the action seems to contain just two "scalars", the tensorial nature of the variables is implicit in their TT nature. Hence the generators must be of the form

$$\begin{aligned} P_0 &= \int d^3x\, H\,, \quad \boldsymbol{P} = -\int d^3x\, p \boldsymbol{\nabla} q \\ \boldsymbol{J} &= \int d^3x \left\{ p(\boldsymbol{r} \times \boldsymbol{\nabla}) q + s\, p \times q \right\} \;=\; \boldsymbol{L} + \boldsymbol{S} \\ \boldsymbol{K} &= \int d^3x\, \boldsymbol{r} H - t \boldsymbol{P} \end{aligned} \tag{3.2}$$

where the spin term $\boldsymbol{S}$ is shorthand for a suitable index contraction scheme (see below). [These generators obey the Poincaré algebra at any time t, in particular one can set $t = 0$ in $\mathbf{K}$. However, since $[\boldsymbol{K}, P_0] \neq 0$ –boosts are time dependent– only $\mathbf{K}(t)$ generates symmetry transformations of the action (3.1).] This "prediction" in turn implies the existence of a set, $T^{\mu\nu}(p,q) = T^{\nu\mu}$, that yields the moments (3.2), and obeys on-shell conservation, $\partial_\mu T^{\mu\nu} = 0$, where $p = \dot{q}$, $\Box q = 0$. Indeed, we can be even more explicit and predict that

$$T^{00} = H + S^{00}\,, \quad T^0_i = -p \partial_i q + s \partial_j (p^{j\ell \cdots} q_{i\ell \ldots}) + S^0_i \tag{3.3}$$

where $S^{0\mu}$ are superpotentials. The only role of T_{ij} is to verify that $\partial_0 T^0_i$ is a spatial divergence, and that is in turn guaranteed by the form (3.3). Furthermore the fundamental commutation relations,

$$i[p_A, q'_B] = [\delta_{AB}(\boldsymbol{r} - \boldsymbol{r}')]^{TT}\,, \tag{3.4}$$

where the right side is TT in each of its variables, guarantee (but non-trivially as we shall note) the Poincaré algebra among the integrated generators (3.2).

4 Spin 1

We now illustrate the above requirements first with a parallel treatment of the Maxwell field before going on to the main, $s > 1$, case. The general procedure, after fixing on a candidate conserved $T^{\mu\nu}$ (all of which differ by a superpotential), is to insert the constraints, remove their Legrange multipliers and fix gauge variables, just leaving the desired gauge invariant pairs (p_A, q_A). [All such gauge fixings are also only a superpotential away from each other.]

Let's begin with the "degenerate" spin 1 case, where no gauge fixing is needed, but constraints still must be solved, as part of the "on-shell" procedure. In first order form, where $(-\boldsymbol{E}_T,\, \boldsymbol{A}^T)$ are

[1] In generic gravitational backgrounds, this invariance is lost and is a symptom of the gravitation interaction problems.

the transverse conjugate variables and $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}^T$,

$$\begin{aligned} T^{00} &= \tfrac{1}{2}\,(\boldsymbol{E}^2 + \boldsymbol{B}^2) = \tfrac{1}{2}\,(p^2 + (\nabla q)^2) - \tfrac{1}{2}\,\partial^2_{ij}(q_i q_j) \equiv H_s + \partial^2_{ij}\Delta^{ij} \\ T^0_i &= (E \times B)_i = -p\partial_i\, q + \partial_j(p, q_i) \equiv T^0_{s\,i} + \partial_j(p_j q_i)\ . \end{aligned} \tag{4.1}$$

Note the spin term in T^0_i, with unit coefficient, and the fact that apart from it, the rest of the T^0_μ are of "scalar", $H_s = \frac{1}{2}(p^2 + (\nabla q)^2)$, $T^0_{s\,i} = -p\partial_i\, q$, form. There is an important lesson here: The Maxwell Poincaré generators are (taking the boosts at $t = 0$ for simplicity)

$$P_\mu = P_\mu(s)\ , \quad K^i = \int d^3x\, x^i T^{00} = K^i(s)\ , \quad \boldsymbol{J} = \boldsymbol{L}(s) + \int d^3x\,(p \times q)\ . \tag{4.2}$$

Apart from the extra spin term in (4.2) they are of pure scalar form. But since the scalar generators "certainly" satisfy the scalar algebra, how does the spin term ever appear in the boost-boost commutator, $[K^i,\, K^j] = \epsilon^{ijk} J_k$, since $[K^i(s), K^j(s)] = \epsilon^{ijk} L_k(s)$ only? The answer is subtle and makes essential use of the fact that our variables are transverse, hence obey the transverse fundamental equal-time commutator,

$$i[p_i, q_{j'}] = [\delta_{ij'}(\boldsymbol{r} - \boldsymbol{r}')]^{TT'} \equiv \delta_{ij'}\delta^3(\boldsymbol{r} - \boldsymbol{r}') + \partial_i\partial_{j'} G(\boldsymbol{r} - \boldsymbol{r}') \tag{4.3}$$

where G is the Coulomb Green Function. If one keeps track of this extra term, then the spin part $\boldsymbol{S}$ duly appears. The lesson is that whenever moments are involved, it is essential to operate in the correct space of transverse (-traceless) tensors.

5 Spin ≥ 2

Let us (at last) turn to the first system of interest, spin 2, *i.e.*, the linearized approximation of GR about (say) flat space (all higher spins behave the same way). There are (as we know from GR) infinitely many candidate $T^{\mu\nu}$ differing by superpotentials, and none is (abelian) gauge invariant. One example is the Landau–Lifshitz complex, which is long but involves only bilinears in the (linearized) $\Gamma^\alpha_{\mu\nu}$; of course it yields the same $T^{\mu\nu}$ as our choice below, up to a superpotential. More useful is simply the quadratic part of the Einstein tensor, which we adopt here; one advantage of choosing it,

$$T_{\mu\nu} \equiv -\tfrac{1}{4}\, G^Q_{\mu\nu} \tag{5.1}$$

is that its conservation is an immediate consequence of the Bianchi identities (at each order in a field expansion) and the on-shell conditions, $G^L_{\mu\nu} = 0$. [The suffixes (Q, L) stand for (quadratic, linear) expression in $h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}$]. Indeed,

$$0 \equiv (D_\mu G^{\mu\nu})_Q \equiv \partial_\mu G^{\mu\nu}_Q + (\Gamma^L G_L)^\nu = \partial_\mu G^{\mu\nu}_Q\ . \tag{5.2}$$

The next advantage, aside from not having to exhibit T^{ij}_Q (since we know there is one!), is that the $G^{0\mu}$ are already the energy-momentum constraints in the full theory. Even a manual calculation is not too difficult, in terms of the only remaining variables $(\dot{h}^{TT}_{ij}, h^{TT}_{ij}) \equiv (p_A, q_A)$, $A = 1, 2$ after constraints and gauge choices are imposed.

One finds the same story as for spin 1,

$$\begin{aligned} T^{00} &\cong \tfrac{1}{2}(p^2 + (\nabla q)^2) \\ T^0_i &\cong -p\partial_i q - 2\partial_j(p_{jk} q_{ik}) \end{aligned} \tag{5.3}$$

where $\cong$ means up to superpotentials. The fundamental (p_A, q'_A) ETC are suitably higher-index transverse-traceless versions of the vector case and one may easily extend (5.3) to arbitrary spins using suitable notation to generalize "$p \times q$". The integrated generators are of the "mostly scalar" Maxwell form as well, except for the "bigger" spin term $\boldsymbol{S}$.

6 Conclusion

We begin with some additional comments: (1) There exist, of course, various representations of the field variables, such as vierbein form involving a non-symmetric $e_{\mu\alpha}$ and a connection $\omega_\mu^{\alpha\beta}$. The action can even be made to resemble a multi-photon system with "internal" index α. But a spin 2 system is *not* merely a photon multiplet, a difference that ruins the gauge invariance of the associated symmetric $T_s^{\mu\nu}(e,\omega)$. In this formulation, the canonical $T_c^{\mu\nu}$ does retain gauge invariance, but to no avail: only symmetric $T^{\mu\nu}$ can define angular momentum. More generally, field redefinitions cannot cure the basic noninvariance problem for $s = 2$, nor *a fortiori* for higher spins. (2) The formulation we have employed here can perhaps be generalized to constant curvature backgrounds [1] but not, as noted, to generic curved spaces. (3) The conditions for Lorentz invariance were realized here by computing the commutation relations among the Poincaré generators. There is also a well-known local criterion for Lorentz invariance in QFT, namely the Dirac-Schwinger ETC.

$$i[T_{00}(\boldsymbol{r}),\ T_{00}(\boldsymbol{r}')] = (T^{0i}\partial_i + T^{0i'}\partial'_i)\delta^3(\boldsymbol{r}-\boldsymbol{r}')\ . \tag{6.1}$$

This form is however, inapplicable here because of the non-manifestly covariant form of our T_μ^0. For example adding Lorentz-variant terms to $T^{\mu\nu}$ is always detected in the integrated ETC, but not in the local form (5.4). This is a difference worth pursuing.

Finally, we emphasize that our new stress tensors, like the covariant ones, are still not suitable for coupling, as currents, to gravity because that requires both invariances to be simultaneously manifest. Indeed, were there a spin 2 gauge invariant tensor, it would imply existence of a consistent cubic self-interacting model of gravity, with abelian invariance and without higher derivatives. Finding consistent dynamical sources of generic $s > 1$ fields seems even more unlikely in any local context.

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